

DIAGRAMS ENCODING GROUP ACTIONS ON Γ -SPACES

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ABSTRACT. We introduce, for any group G , a category $G\Gamma$ such that diagrams $G\Gamma \rightarrow \mathcal{SSets}$ satisfying a Segal condition correspond to infinite loop spaces with a G -action.

1. INTRODUCTION

In [1] and [3], we looked at diagrammatic ways to approach simple algebraic structures as diagrams satisfying some kind of Segal condition. The terminology comes from the fact that such conditions were first investigated by Segal [6]. In [3], we show that diagrams $X: \Delta^{op} \rightarrow \mathcal{SSets}$ such that $X_0 = \Delta[0]$ can be regarded as simplicial monoids when the Segal condition holds either strictly or up to homotopy. In contrast, Segal shows that for diagrams $Y: \Gamma^{op} \rightarrow \mathcal{SSets}$ with $Y_0 = \Delta[0]$, those that satisfy the Segal condition strictly are equivalent to simplicial abelian monoids, whereas those satisfying it only up to homotopy, simply called Γ -spaces, can be regarded as infinite loop spaces, at least with an additional group-like condition.

In [1], we built on work of Bousfield [5] to encode group and abelian group structures, not by changing the diagram shape, but by modifying the Segal condition to one we call a Bousfield-Segal condition.¹ In particular, for Γ -spaces this condition corresponds exactly to the group-like condition required for an infinite loop space.

Here, we use an approach from [4] to form categories built from multiple copies of Γ so that its diagrams of simplicial sets satisfying the up-to-homotopy Segal condition correspond to infinite loop spaces with a G -action, for a specified discrete group G . In future work, we hope to treat the more general case, where the group G varies and can be taken to be a simplicial group, analogously to what we do in [4].

2. BACKGROUND

In this section we give a review of Segal's category Γ and some relevant results. By \mathcal{SSets} , we denote the category of simplicial sets, or functors $\Delta^{op} \rightarrow \mathcal{Sets}$ with the model structure equivalent to the usual model category of topological spaces. Here the simplicial indexing category Δ has objects the finite ordered sets and morphisms are the order-preserving maps.

2.1. The category Γ and Segal maps. We begin with the original definition of the category Γ , as given by Segal in [6]. Its objects are representatives of isomorphism classes of finite sets, and a morphism $S \rightarrow T$ is given by a map $\theta: S \rightarrow \mathcal{P}(T)$

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¹Note the change-of-diagram approach in that paper is incorrect; see [2].

such that $\theta(\alpha)$ and $\theta(\beta)$ are disjoint whenever $\alpha \neq \beta$. (Here $\mathcal{P}(T)$ is the power set of the set T .) The opposite category Γ^{op} has the following description of its own: the category with objects $\mathbf{n} = \{0, 1, \dots, n\}$ for $n \geq 0$ and morphisms $\mathbf{m} \rightarrow \mathbf{n}$ such that $0 \mapsto 0$.

In Γ^{op} , there are maps $\varphi_{n,k}: \mathbf{n} \rightarrow \mathbf{1}$ for any $1 \leq k \leq n$ given by, for any $0 \leq i \leq n$,

$$\varphi_{n,k}(i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

Given any functor $X: \Gamma^{op} \rightarrow \mathcal{S}Sets$, we get induced maps $\varphi_{n,k}: X(\mathbf{n}) \rightarrow X(\mathbf{1})$. The disjoint union

$$\varphi_n = \coprod_{k=1}^n \varphi_{n,k}$$

is called a *Segal map*.

Definition 2.1. A Γ -space X is a functor $\Gamma^{op} \rightarrow \mathcal{S}Sets$ such that $X(\mathbf{0}) \cong \Delta[0]$ and the Segal map $\varphi_n: X(\mathbf{n}) \rightarrow (X(\mathbf{1}))^n$ is a weak equivalence of simplicial sets for each $n \geq 2$. If $X(\mathbf{0}) \cong \Delta[0]$ and if each Segal map $\varphi_n: X(\mathbf{n}) \rightarrow (X(\mathbf{1}))^n$ is an isomorphism, then X is a *strict Γ -space*.

Note that our definition differs from the original, in that even for non-strict Γ -spaces we require that $X(\mathbf{0})$ be a single point, rather than simply equivalent to one.

The following two results are due to Segal [6].

Proposition 2.2. *The category of strict Γ -spaces is equivalent to the category of simplicial abelian monoids.*

Proposition 2.3. *The category of Γ -spaces X such that $\pi_0(X(\mathbf{1}))$ is a group is equivalent to the category of infinite loop spaces.*

Some constructions for Γ can also be applied to Δ . Segal defines a functor $\Delta \rightarrow \Gamma$ as follows. The object $[n]$ is sent to \mathbf{n} for each $n \geq 0$, and a map $f: [m] \rightarrow [n]$ is sent to the map $\theta: \mathbf{m} \rightarrow \mathbf{n}$ given by $\theta(i) = \{j \in \mathbf{n} \mid f(i-1) < j \leq f(i)\}$.

We can also define Segal maps for simplicial spaces, or functors $X: \Delta^{op} \rightarrow \mathcal{S}Sets$; here also we restrict to the case where $X_0 \cong \Delta[0]$. The Segal map $X_n \rightarrow (X_1)^n$ is induced by the maps $\alpha^{n,k}: [1] \rightarrow [n]$ in Δ for each $0 \leq k \leq n-1$, where $\alpha^{n,k}(0) = k$ and $\alpha^{n,k}(1) = k+1$. Then applying the functor $\Delta \rightarrow \Gamma$, these maps $\alpha^{n,k}$ are sent to $\varphi_{n,k}^{op}$.

2.2. Bousfield-Segal maps. Following an idea of Bousfield [5], in Δ we define the maps $\gamma^{n,k}: [1] \rightarrow [n]$ given by $0 \mapsto 0$ and $1 \mapsto k+1$ for all $0 \leq k < n$. Restricting to the case where $X_0 \cong \Delta[0]$, we can define the *Bousfield-Segal map* $X_n \rightarrow (X_1)^n$ induced by these maps. When such an X satisfies the condition that the Bousfield-Segal maps are weak equivalences of simplicial sets for all $n \geq 2$, we call it a *Bousfield-Segal group*; the name is justified by the fact that it is equivalent to a simplicial group [1]. A comparison to the usual Segal condition is given in figure 1; the idea is to define a group in terms of the binary operation $a, b \mapsto ab^{-1}$.

Translating the maps $\gamma^{n,k}$ from Δ to Γ , we get maps $\delta^{n,k}: \mathbf{1} \rightarrow \mathbf{n}$, one for each $1 \leq k \leq n$.

Definition 2.4. A *strict Bousfield Γ -space* is a functor $X : \Gamma^{op} \rightarrow \mathcal{SSets}$ such that $X(\mathbf{0}) \cong \Delta[0]$ and the maps $X(\mathbf{n}) \rightarrow X(\mathbf{1})^n$ induced by the maps $\delta^{n,k}$ for all $1 \leq k \leq n$ are isomorphisms for all $n \geq 2$. Similarly, a (*homotopy*) *Bousfield Γ -space* has $X(\mathbf{0}) \cong \Delta[0]$ and the maps $X(\mathbf{n}) \rightarrow X(\mathbf{1})^n$ weak equivalences of simplicial sets for $n \geq 2$.

Proposition 2.5. [1, 7.2] *The category of strict Bousfield Γ -spaces is equivalent to the category of simplicial abelian groups.*

The following result was stated in [1]; using the fact that the Bousfield-Segal condition gives a group structure, the group-like condition of Segal is satisfied.

Proposition 2.6. *The category of Bousfield Γ -spaces is equivalent to the category of infinite loop spaces.*

2.3. Algebraic theories. For strict Γ -spaces and Bousfield Γ -spaces, one method of comparison to their respective algebraic structures is via the machinery of algebraic theories.

Definition 2.7. An *algebraic theory* \mathcal{T} is a small category with finite products and objects denoted T_n for $n \geq 0$. For each n , T_n is equipped with an isomorphism $T_n \cong (T_1)^n$. Note in particular that T_0 is the terminal object in \mathcal{T} .

Definition 2.8. Given an algebraic theory \mathcal{T} , a (*strict simplicial*) \mathcal{T} -*algebra* A is a product-preserving functor $A : \mathcal{T} \rightarrow \mathcal{SSets}$.

Here, “product-preserving” means that for each $n \geq 0$ the canonical map

$$A(T_n) \rightarrow A(T_1)^n,$$

induced by the n projection maps $T_n \rightarrow T_1$, is an isomorphism of simplicial sets. In particular, $A(T_0)$ is the one-point simplicial set $\Delta[0]$. For a given algebraic theory \mathcal{T} , we denote by $\mathcal{Alg}^{\mathcal{T}}$ the category of \mathcal{T} -algebras.

The proofs of Propositions 2.2 and 2.5 can be established by comparing the categories of simplicial abelian monoids and simplicial abelian groups to the categories of algebras over the theories $\mathcal{T}_{\mathcal{AM}}$ of abelian monoids and $\mathcal{T}_{\mathcal{AG}}$ of abelian groups, respectively. Each of these categories has as objects the finitely generated free objects in the appropriate category.

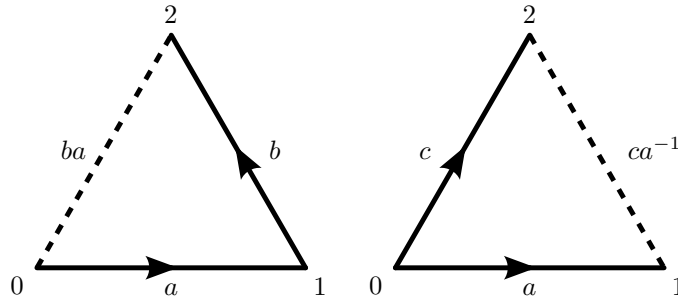


FIGURE 1. Left: Segal condition, Right: Bousfield-Segal condition

3. ACTIONS OF A FIXED GROUP ON Γ -SPACES

In this section, we give a diagrammatic description of infinite loop spaces with a G -action, for a given group G . The diagram we give is essentially a wedge product of copies of Γ , indexed by the elements of the group G . We assume throughout that a discrete group G is fixed.

Generalizing the case of strict Γ -spaces, we begin by considering abelian monoids with a G -action. There is an algebraic theory $\mathcal{T}_{G\mathcal{AM}}$ of abelian monoids with G -action; it is the full subcategory of the category of abelian monoids with objects, for each $n \in \mathbb{N}$, given by $G \times F_n$, where F_n denotes the free abelian monoid on n generators.

Define $G\Gamma^{op}$ to be the category with objects $\mathbf{n}_G = \bigvee_{g \in G} \mathbf{n}_g$, where $\mathbf{n}_g = \{0_g, 1_g, \dots, n_g\}$. The morphisms are generated by:

- morphisms $\bigvee_{g \in G} f: \bigvee_{g \in G} \mathbf{n}_g \rightarrow \bigvee_{g \in G} \mathbf{m}_g$ where each $f: \mathbf{n}_g \rightarrow \mathbf{m}_g$ is given by the same morphism $f: \mathbf{n} \rightarrow \mathbf{m}$ of Γ^{op} , and
- automorphisms given by a G -action, so, for $g \in G$, $g \cdot: \mathbf{n}_G \rightarrow \mathbf{n}_G$ given by $g \cdot k_h = k_{gh}$.

Example 3.1. Consider the case where $G = \mathbb{Z}/2 = \{0, 1\}$. Then the objects $\mathbf{n}_{\mathbb{Z}/2}$ can be thought of as pointed sets

$$\{n_0, \dots, 2_0, 1_0, 0, 1_1, 2_1, \dots, n_1\}.$$

The action of $\mathbb{Z}/2$ sends each k_0 to k_1 and vice versa.

Consider functors $X: G\Gamma^{op} \rightarrow \mathcal{S}Sets$ such that $X(\mathbf{0}_G) \cong \Delta[0]$. The Segal maps for Γ induce Segal maps $X(\mathbf{n}_G) \rightarrow (X(\mathbf{1}_G))^n$ for $G\Gamma$, where the behavior on each copy of Γ in $G\Gamma$ is the same. Similarly, the Bousfield-Segal maps for Γ induce Bousfield-Segal maps for $G\Gamma$.

The proof of the following proposition generalizes the one given in [1, 7.1].

Proposition 3.2. *The category of functors $X: G\Gamma^{op} \rightarrow \mathcal{S}Sets$ satisfying the strict Segal condition is equivalent to the category of simplicial abelian monoids equipped with a G -action.*

Proof. First recall that the category of simplicial abelian monoids with a group action is equivalent to the category $\mathcal{Alg}^{\mathcal{T}_{G\mathcal{AM}}}$ of strict algebras over $\mathcal{T}_{G\mathcal{AM}}$, so it suffices to establish an equivalence with $\mathcal{Alg}^{\mathcal{T}_{G\mathcal{AM}}}$.

In $G\Gamma^{op}$, there are projection maps $p_{n,i,G}: \mathbf{n}_G \rightarrow \mathbf{1}_G$ where $p_{n,i,G}(k_g) = 1_g$ if $k = i$ and 0 otherwise. The natural functor $G\Gamma^{op} \rightarrow \mathcal{T}_{G\mathcal{AM}}$, given by $\mathbf{n}_G \mapsto G \times F_n$, preserves these projection maps.

Given a strict $G\Gamma$ -space X , it is determined by each simplicial set $X(\mathbf{n}_G)$, the projection maps $X(\mathbf{n}_G) \rightarrow X(\mathbf{1}_G)$, and the map $X(\mathbf{2}_G) \rightarrow X(\mathbf{1}_G)$ which is the image of the map $\mathbf{2}_G \rightarrow \mathbf{1}_G$ given by $0 \mapsto 0$ and $1_g, 2_g \mapsto 1_g$ for each $g \in G$. In particular, by induction the map $X(\mathbf{2}_G) \rightarrow X(\mathbf{1}_G)$ induces all maps $X(\mathbf{n}_G) \rightarrow X(\mathbf{1}_G)$ arising from the projections $\mathbf{n}_G \rightarrow \mathbf{1}_G$ given by $0 \mapsto 0$ and $i_g \mapsto 1_g$ for each $1 < i \leq n$ and $g \in G$. Then the structure of a strict $G\Gamma$ -space gives the space $X(\mathbf{1}_G)$ the structure of an abelian monoid (with multiplication map given by $X(\mathbf{2}_G) \rightarrow X(\mathbf{1}_G)$ as above), with a G -action given by the map $G \times X(\mathbf{1}_G) \rightarrow X(\mathbf{1}_G)$ induced from $G \times \mathbf{1}_G \rightarrow \mathbf{1}_G$ defined by $(g, 1_h) \mapsto 1_{gh}$.

In particular, $X(\mathbf{1}_G)$ defines a $\mathcal{T}_{G\mathcal{AM}}$ algebra tX which has the same spaces $X(\mathbf{n}_G)$ at level n , and projection maps preserved from X . Restricting tX back to

a $G\Gamma$ -space results in the original X . Denoting this restriction map by F , we have shown that the functors t and F are inverses. \square

Corollary 3.3. *The category of functors $X: G\Gamma^{op} \rightarrow \mathcal{S}Sets$ satisfying the strict Bousfield-Segal condition is equivalent to the category of simplicial abelian groups equipped with a G -action.*

Proposition 3.4. *A functor $X: G\Gamma^{op} \rightarrow \mathcal{S}Sets$ satisfying the (homotopy) Segal condition determines a spectrum with a G -action.*

Proof. Suppose that X is a $G\Gamma$ -space. Then $X(\mathbf{0}_G)$ is a point and the Segal maps $X(\mathbf{n}_G) \rightarrow X(\mathbf{1}_G)^n$ are weak equivalences for $n \geq 2$. To construct the classifying space of a Γ -space X , Segal first defines a bi- Γ -space $\widehat{X}(\mathbf{m}, \mathbf{n}) = X(\mathbf{m} \wedge \mathbf{n})$. Then the classifying space BX is obtained by composing the adjoint of the composite map

$$\Delta^{op} \times \Gamma^{op} \rightarrow \Gamma^{op} \times \Gamma^{op} \rightarrow \mathcal{S}Sets$$

with ordinary geometric realization of a simplicial space to get

$$\Gamma^{op} \rightarrow \mathcal{S}Sets^{\Delta^{op}} \rightarrow \mathcal{S}Sets.$$

Since we require that $X(\mathbf{0})$ be a point, we should verify that this condition still holds for the classifying space. The space $BX(\mathbf{0})$ is the geometric realization of the functor $\mathbf{n} \mapsto X(\mathbf{0} \wedge \mathbf{n})$. Since $X(\mathbf{0} \wedge \mathbf{n}) = X(\mathbf{0})$, we have that $BX(\mathbf{0})$ is the geometric realization of the constant simplicial space of a point.

Now we consider the case at hand, where we have functors $\Delta^{op} \rightarrow \Gamma^{op}$ and $e: \Gamma^{op} \rightarrow G\Gamma^{op}$. If X is a $G\Gamma$ -space, we can perform the same construction as above to get a bi- $G\Gamma$ -space defined by $(\mathbf{n}_G, \mathbf{m}_G) \mapsto X(\mathbf{n}_G \wedge \mathbf{m}_G)$, which we denote by \widehat{X} . Observing that $\widehat{e^*X} = (e \times e)^*\widehat{X}$, we have the commutative diagram

$$\begin{array}{ccc} \Delta^{op} \times \Gamma^{op} & \longrightarrow & \Gamma^{op} \times \Gamma^{op} \\ \downarrow & & \downarrow \\ \Delta^{op} \times G\Gamma^{op} & \longrightarrow & G\Gamma^{op} \times G\Gamma^{op} \end{array} \quad \begin{array}{c} \searrow \widehat{e^*X} \\ \nearrow \widehat{X} \end{array} \quad \begin{array}{c} \\ \\ \end{array} \mathcal{S}Sets.$$

Taking adjoints, we have the commutative diagram

$$\begin{array}{ccc} \Gamma^{op} & & \\ \downarrow & \searrow & \\ & \mathcal{S}Sets^{\Delta^{op}} & \xrightarrow{|\cdot|} \mathcal{S}Sets. \\ \uparrow & \nearrow & \\ G\Gamma^{op} & & \end{array}$$

We then see that $B(e^*X)$ naturally has the structure of a $G\Gamma$ -space, and we denote it BA .

Iterating this construction, we obtain a sequence of spaces

$$X(\mathbf{1}_G), BX(\mathbf{1}_G), B^2X(\mathbf{1}_G), \dots$$

which is a spectrum since X is a Γ -space. Each $B^n X(\mathbf{1}_G)$ here is also equipped with a G -action, so it remains to show that the spectrum structure maps preserve this action.

In the ordinary case of Γ -spaces, Segal shows that the 1-skeleton of X , $\mathrm{sk}_1|X|$, is homotopy equivalent to $\Sigma X(\mathbf{1})$; the argument can be given using a diagram such as the following:

$$\begin{array}{ccccc}
 (* \amalg (\Delta^1 \times X_1))/\sim & \longleftarrow & ((\Delta^0 \times X_0) \amalg (\Delta^1 \times X_1))/\sim & & \\
 \downarrow = & & \downarrow = & & \\
 \Sigma X(\mathbf{1}) & \xleftarrow{\quad \simeq \quad} & \mathrm{sk}_1|X| & \xrightarrow{\quad} & |X|.
 \end{array}$$

Since $X(\mathbf{0})$ is contractible, the upper horizontal map is a homotopy equivalence, and therefore so is the indicated lower horizontal map. The spectrum structure map is given by choosing a homotopy inverse to this map and postcomposing with the inclusion $\mathrm{sk}_1|X| \rightarrow |X|$.

In our case, we have imposed the more restrictive condition that $X(\mathbf{0}_G) = *$. It follows that $\Sigma X(\mathbf{1}_G)$ is isomorphic to $\mathrm{sk}_1|X|$ (not just weakly equivalent), so the structure map is given by the inclusion $\Sigma X(\mathbf{1}_G) \cong \mathrm{sk}_1|X| \rightarrow |X| = BX(\mathbf{1}_G)$, which is G -equivariant. \square

Corollary 3.5. *If $X : G\Gamma^{op} \rightarrow \mathcal{S}Sets$ satisfies the Bousfield-Segal condition, then $X(\mathbf{1})$ is an infinite loop space with G -action.*

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